SUBJECT: DECISION SCIENCE

# Module 4: Business in the Factor Market

Comprehensive University-Level Study Notes • Units 16 – 22

## 16 Decision Theory- Decisions under Certainty, Uncertainty

### Introduction to Decision Theory

Decision Theory is a systematic, quantitative framework used to evaluate and make optimal choices among multiple alternative courses of action when faced with varying degrees of future ambiguity. In management analysis, a decision model consists of three basic components: **Courses of Action (Acts)** represented as alternative options controlled by the decision-maker, **States of Nature (Events)** representing future environmental conditions outside the decision-maker's control, and **Payoffs**, which measure the quantitative net return resulting from each specific combination of an act and a state of nature.

### Decisions Under Certainty

A decision situation operates under certainty when the decision-maker possesses absolute, perfect knowledge regarding which specific state of nature will occur. In this deterministic environment, each course of action leads to exactly one predictable payoff outcome. Common operational tools utilized to resolve decisions under certainty include Linear Programming, basic break-even allocation models, and deterministic inventory EOQ calculations.

### Decisions Under Uncertainty

Decisions operate under uncertainty when multiple future states of nature are possible, but the decision-maker has absolutely no empirical data or statistical historical basis to assign probabilities to those events. To resolve choices in this highly ambiguous scenario, managers utilize subjective, structured decision criteria:

- **1. Maximax Criterion (Optimistic Rule):** The decision-maker assumes the best possible environment will occur. The algorithm selects the maximum payoff out of the maximum available payoffs for each course of action. Suitable for risk-seeking entities.
- **2. Maximin Criterion (Pessimistic Rule):** The decision-maker acts conservatively to protect the organization against worst-case scenarios. The algorithm identifies the minimum possible payoff for each alternative and selects the course of action that yields the maximum out of those minimums. Suitable for risk-averse managers.
- **3. Minimax Regret Criterion (Savage Rule):** Focuses on minimizing the maximum potential "opportunity loss" or regret. Regret is computed for each cell by subtracting the actual payoff from the maximum available payoff under that specific state of nature. The algorithm isolates the maximum regret for each act and selects the option that minimizes this maximum penalty.
- **4. Laplace Criterion (Equally Likely Rule):** Assumes that since probabilities are completely unknown, all states of nature carry an equal likelihood of occurrence. It computes the simple mathematical arithmetic average of payoffs for each course of action and selects the act with the highest average.
- **5. Hurwicz Criterion (Criterion of Realism):** Strikes a formal balance between extreme optimism and extreme pessimism. It utilizes an assigned index of optimism, denoted as alpha ($\alpha$), where $0 \leq \alpha \leq 1$. The expected value for each alternative action is calculated using the weighted equation:

  $$Weighted\ Value = \alpha \times (Maximum\ Payoff) + (1 - \alpha) \times (Minimum\ Payoff)$$

  The course of action with the highest calculated weighted index value is chosen.



## 17 Risk and Conflict, Payoff Matrix, Decision Tree

### Decisions Under Risk and Conflict

A decision under **Risk** occurs when multiple states of nature can emerge, but the decision-maker possesses reliable historical data or objective parameters to assign explicit probabilities to each event. The primary analytical tool deployed in risk environments is the **Expected Monetary Value (EMV)** engine. For each course of action, the EMV is calculated by summing the products of the conditional payoffs and their respective probabilities:

$$EMV(Act\ A_i) = \Sigma\ [\ Payoff(A_i, S_j) \times Probability(S_j)\ ]$$

The alternative that yields the maximum EMV is selected. Related parameters include **Expected Opportunity Loss (EOL)**, which minimizes expected regret, and **Expected Value of Perfect Information (EVPI)**, which measures the maximum economic value a manager should pay to eliminate uncertainty entirely (**EVPI = EVWPI – Maximum EMV**).

Decisions under **Conflict** involve competitive scenarios where the outcomes depend heavily on the rational actions of opposing, intelligent adversaries. This specific environment is resolved using Game Theory modeling.

## The Payoff Matrix Structure

A Payoff Matrix is a structured, two-dimensional grid format used to display the intersection values of decision profiles. The rows of the grid explicitly list the alternative courses of action (Acts) available to the manager, while the columns present the mutually exclusive future environmental states of nature. Each cell contains the specific quantitative payoff generated by that intersecting pair.

## Decision Tree Analysis

A Decision Tree is a chronological, graphical diagram that models a multi-stage decision problem as a series of sequential branching lines. It provides a visual framework for evaluating choices that lead to downstream events and subsequent future choices.

**Core Components of a Decision Tree:**

- **Decision Nodes (Represented as Squares):** Points where the decision-maker must actively choose a specific path from available alternatives.
- **Chance / Event Nodes (Represented as Circles):** Points where an uncertain state of nature occurs, branching out into paths weighted by explicit probabilities.
- **Terminal Branches:** The end points of the branches displaying the final conditional payoff values.

[Image of a Decision Tree diagram]

### The Backward Induction Evaluation Engine

To evaluate a multi-stage decision tree, decision scientists utilize the **Rollback (Backward Induction)** technique, working from right to left across the diagram layout:

1. Begin at the terminal branches on the extreme right of the tree schema.
2. At each chance node (circle), compute the Expected Monetary Value (EMV) by multiplying branch payoffs by their probabilities and summing the results. Collapse the node down to this single EMV value.
3. At each decision node (square), compare the values of all incoming branches, select the single most profitable path, and prune (cross out) the sub-optimal alternative branches.
4. Continue rolling backward until reaching the root decision node on the far left. The value remaining at the root node indicates the maximum expected profit path.

## 18 Game Theory - Concept and Definition

### Concept and Definition of Game Theory

Game Theory is a specialized branch of mathematical optimization focused on the study of competitive situations where the final outcome depends entirely on the combined strategic choices of multiple rational, intelligent adversaries. Formally defined, a "game" is a mathematical model of conflict or cooperation among independent actors who make decisions designed to maximize their individual payoffs while anticipating the counter-moves of their competitors.

### Fundamental Underlying Assumptions of Game Structures

For a competitive situation to be modeled validly under game theory rules, it must satisfy the following parameters:

- **Rational Behavior:** Every player acts logically with the sole objective of maximizing their own gains or minimizing their personal losses, assuming their opponent will act with equal intelligence.
- **Finite Strategies:** Each participant has access to a known, finite number of distinct alternative courses of action (strategies) before the game starts.
- **Simultaneous Choices:** Players select their specific strategies simultaneously and in complete secrecy, meaning no player knows their opponent's choice before finalizing their own.

- **Perfect Matrix Information:** The complete payoff matrix, including all possible net returns for every combinations of strategies, is fully known to all players in advance.

### Core Classifications and Key Terminologies

Games are classified based on the number of players and the structural nature of the payouts:

1. **Two-Person Zero-Sum Game:** A competitive structure involving exactly two players where the economic gain of Player A is exactly equal to the economic loss of Player B. The algebraic sum of net payoffs after any play is precisely zero. Often called a rectangular game when mapped as a matrix.
2. **Pure Strategy:** A deterministic situation where a player chooses a single, fixed course of action every time they play, completely abandoning all alternative options.
3. **Mixed Strategy:** A probabilistic approach where a player introduces uncertainty for their opponent by mixing multiple strategies over a series of plays according to fixed, pre-calculated probabilities.

## 19 Solution Methods of Pure Strategy games (with Saddle Point)

### The Concept of a Saddle Point

A Saddle Point represents a state of stable equilibrium in a pure strategy game. It occurs at a matrix coordinate position where the security policies of both players intersect perfectly. Mathematically, a payoff matrix possesses a saddle point when the maximum of the row minimums (Maximin) exactly equals the minimum of the column maximums (Minimax):

$$Maximin\ Value = Minimax\ Value$$

When a saddle point is present, neither player has any economic incentive to change their strategy unilaterally, as doing so would worsen their individual payoff. The value located at this intersecting matrix cell is defined as the absolute **Value of the Game (V)**. If **V = 0**, the game is strictly **Fair**; if **V ≠ 0**, the game is biased toward one of the players.

## The Step-by-Step Saddle Point Algorithm

To solve a pure strategy game, deploy the following algorithmic matrix evaluation sequence:

> **Step 1: Compute Row Minima:** Examine each individual row of the payoff matrix independently. Identify the absolute minimum element in each row and record it in a new column on the right margin.
>
> **Step 2: Isolate Maximin Value:** Scan the recorded row minimum values and select the single largest value among them. Underline or circle this number; it represents the **Maximin** value.
>
> **Step 3: Compute Column Maxima:** Examine each individual column of the matrix independently. Identify the absolute maximum element in each column and record it in a new row along the bottom margin.
>
> **Step 4: Isolate Minimax Value:** Scan the recorded column maximum values and select the single smallest value among them. Mark this number; it represents the **Minimax** value.
>
> **Step 5: Verify Equilibrium:** Compare the isolated Maximin and Minimax values. If they are identical, that specific cell coordinate is the Saddle Point. The corresponding row and column indicate the optimal pure strategies for Player A and Player B respectively.

[Image of a Two-Person Zero-Sum Game Payoff Matrix with Saddle Point]

## Practical Illustration Matrix Problem

Consider the following 3x3 game matrix representing the payoffs for Player A (Row Player):

| Player A Strategies | Player B: Strategy B1 | Player B: Strategy B2 | Player B: Strategy B3 | Row Minima |
|---|---|---|---|---|
| **Strategy A1** | 8 | 2 | 6 | 2 |
| **Strategy A2** | 5 | 3 | 4 | **3 (Maximin)** |

| Player A Strategies | Player B: Strategy B1 | Player B: Strategy B2 | Player B: Strategy B3 | Row Minima |
|---|---|---|---|---|
| **Strategy A3** | 9 | 1 | 0 | 0 |
| **Column Maxima** | **9** | **3 (Minimax)** | **6** | **Saddle Point = 3** |

**Analysis Output:** The minimums of Row 1, 2, and 3 are 2, 3, and 0 respectively. The maximum of these values is **3** (Row 2). The maximums of Column 1, 2, and 3 are 9, 3, and 6 respectively. The minimum of these values is **3** (Column 2). Since **Maximin (3) = Minimax (3)**, a saddle point exists at coordinate (A2, B2). The optimal choice is Strategy A2 for Player A and Strategy B2 for Player B. The final Value of the Game is **V = 3**.

## 20 Theory of Replacement: Introduction

### Core Meaning and Philosophy

The Theory of Replacement is a vital operational decision-making framework focused on determining the exact optimal point in time to replace capital assets, machinery, or individual components. All physical assets degrade or lose structural value over time due to wear and tear, operational usage, passage of time, or technological obsolescence. The core goal of replacement analysis is to balance the initial purchase capital expenditure against rising maintenance and operating costs over the asset's lifecycle, identifying the timing that minimizes the total long-term average cost of operations.

### Primary Triggers for Asset Replacement

In industrial systems management, assets require replacement due to two distinct physical phenomena:

- **Gradual Efficiency Deterioration:** Assets that wear down progressively through continuous operation (e.g., transport vehicles, factory boilers, manufacturing equipment). As the asset ages, its operating and maintenance costs surge, its fuel efficiency plummets, and its scrap resale value drops.
- **Sudden Total Failure:** Atomic elements that do not deteriorate gradually but operate at 100% capacity until they fail completely and abruptly (e.g., electrical lightbulbs, electronic microprocessors, circuit components). These require specialized probabilistic group replacement models.

- **Obsolescence:** Items that remain functionally operational but become economically uncompetitive due to the introduction of advanced, high-efficiency technologies into the market.

# 21 Replacement Models

## Classifications of Replacement Models

Replacement scenarios are systematically structured into specific mathematical models based on the asset's behavioral characteristics and the time value of capital:

> **The Replacement Model Taxonomy Matrix:**
> • **Model Type I: Replacement of items whose operational efficiency deteriorates gradually over time, assuming the value of money remains constant** (handled analytically via simple average annual cost minimization structures).
> • **Model Type II: Replacement of items that deteriorate gradually over time, where the time value of money changes** (handled using present value discount factors and capital cost capitalizations).
> • **Model Type III: Replacement of items that fail suddenly and completely** (evaluated using individual vs. collective group replacement scheduling frameworks).

## Economic Cost Trade-off Mechanics

The mathematical center of any replacement model is an economic trade-off curve. In early years, the average capital cost per year is high because the initial purchase price is spread over few years, while annual maintenance costs are low. As time passes, the annualized capital cost drops, but maintenance costs rise. Summing these values yields a U-shaped total average cost curve. The lowest point on this curve identifies the optimal replacement age.

# 22 Replacement of items that deteriorates gradually (value of money does not change with time)

## The Mathematical Framework and Derivation

This model addresses assets that degrade progressively over time, assuming that inflation and the time value of money are zero. Let **C** equal the initial procurement capital cost of the asset, and **S(t)** equal the scrap (resale) value of the asset after *t* years of operation. The total depreciation cost incurred if the item is held for *n* years is represented as **C – S(n)**. Let **M(t)** represent the operational maintenance cost incurred explicitly during year *t*.

The total cumulative operating cost incurred over an *n*-year horizon is the summation of maintenance costs: $\boldsymbol{\Sigma_{t=1}^{n} M(t)}$. Therefore, the total cumulative cost of owning and running the asset for *n* years is defined as:

$$TC(n) = C - S(n) + \Sigma_{t=1}^{n} M(t)$$

To find the optimal replacement time, we must minimize the **Average Annual Cost**, denoted as **ATC(n)**, which divides total cost by the number of years held:

$$ATC(n) = [\ C - S(n) + \Sigma_{t=1}^{n} M(t)\ ] / n$$

## The Replacement Decision Criterion Rules

Using mathematical differentiation, the optimal replacement age is reached when the average annual cost is at its minimum. This leads to two strict rules used to audit asset lifecycles:

1. Replace the asset at the end of year *n* if the maintenance cost for the next year, **M(n+1)**, is **strictly greater** than the average annual cost of the current year, **ATC(n)**.
2. Retain the asset if the maintenance cost for the next year, **M(n+1)**, is **less than** the current average annual cost, **ATC(n)**.

## Comprehensive Numerical Tabular Example

Consider a machine with an initial cost **C = ₹60,000** and a constant scrap value **S = ₹0**. The maintenance costs per year are detailed below:

| Year (n) | Maintenance Cost M(n) | Cumulative Maintenance ΣM(n) | Total Cost TC(n) = C − S + ΣM(n) | Average Annual Cost ATC(n) = TC(n)/n |
|---|---|---|---|---|
| 1 | 4,000 | 4,000 | 64,000 | 64,000 |
| 2 | 5,000 | 9,000 | 69,000 | 34,500 |
| 3 | 7,000 | 16,000 | 76,000 | 25,333 |
| 4 | 10,000 | 26,000 | 86,000 | 21,500 |
| 5 | 14,000 | 40,000 | 1,00,000 | **20,000 (Minimum)** |
| 6 | 22,000 | 62,000 | 1,22,000 | 20,333 |

**Analysis Output:** Reviewing the final column, the Average Annual Cost drops continuously, hitting its absolute minimum value of **₹20,000 at the end of Year 5**, before rising in Year 6 (₹20,333). Let's apply our decision rules: the maintenance cost for Year 6 is **M(6) = ₹22,000**, which is strictly greater than the current average cost **ATC(5) = ₹20,000**. Therefore, the optimal policy is to **replace the machine at the end of Year 5** to minimize long-term operational costs.



End of Module 4 • Subject: Decision Science